

Exposing points on the boundary of a strictly pseudoconvex or a locally convexifiable domain of finite 1-type

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ABSTRACT: We show that for any bounded domain $\Omega \subset \mathbb{C}^n$ of finite 1-type $2k$ which is locally convexifiable at $p \in b\Omega$, having a Stein neighborhood basis, there is a biholomorphic map $f : \bar{\Omega} \rightarrow \mathbb{C}^n$ such that $f(p)$ is a global extreme point of type $2k$ for $f(\bar{\Omega})$.

1 Introduction

In this paper we consider bounded locally convexifiable domains Ω in \mathbb{C}^n of finite 1-type whose closures $\bar{\Omega}$ admit a Stein neighborhood basis. Here the term "locally convexifiable near $p \in b\Omega$ " means that there are a neighborhood V of p and a one-to-one holomorphic map $\Phi : V \rightarrow \mathbb{C}^n$ such that $\Phi(\Omega \cap V)$ is convex. For the notion of finite type we refer to [2]. Strongly pseudoconvex domains are examples of such domains. We will first prove the following:

Theorem 1.1 *Let $\Omega \subset \mathbb{C}^n$ be a bounded domain which is locally convexifiable and has finite type $2k$ near a point $p \in b\Omega$. Assume further that $b\Omega$ is \mathcal{C}^∞ -smooth near p , and that $\bar{\Omega}$ has a Stein neighborhood basis. Then there exists a holomorphic embedding $f : \bar{\Omega} \rightarrow \mathbb{B}_k^n$, where $\mathbb{B}_k^n = \{z \in \mathbb{C}^n : |z_n|^2 + \|z'\|^{2k} < 1\}$, such that*

1. $f(p) = e_n = (0, \dots, 0, 1)$, and
2. $\{z \in \bar{\Omega} : f(z) \in b\mathbb{B}_k^n\} = \{p\}$.

If $k = 1$, i.e., if $b\Omega$ is strongly pseudoconvex near p , it is enough to assume that $b\Omega$ is \mathcal{C}^2 -smooth near p .

Definition 1.2 *Let $\Omega \subset \mathbb{C}^n$ be a domain and let $p \in b\Omega$ be a point. We say that p is a globally exposed $2k$ -convex point if there exists an affine linear map f as in the previous theorem.*

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One of our motivations for proving this theorem is the special case of strictly pseudoconvex domains. In this case the theorem answers a question posed by Fusheng Deng (private communication), and it is a step to study squeezing functions on bounded strongly pseudoconvex domains (see [3]).

Another motivation consists in the construction of a \mathcal{C}^∞ family of local holomorphic support functions $\hat{S}(z, \zeta) \in \mathcal{C}^\infty(\mathbb{C}^n \times \partial D)$ for locally convexifiable domains of finite type $2k$ with Stein neighborhood basis as explained in [4]. It has been asked several times whether these support functions can always be chosen such that they are globally supporting for the given domains.

However, it has to be asked in which way precisely this question should be answered. As far as we can see, there are at least the following different possibilities, each of them leading to quite different answers:

1. Our original support surfaces are defined only locally. The danger might be, for instance, that after a while they fall back into the inside of the domain or, at least, become tangent at certain points, that are further away. However, this danger can be avoided by applying a simple standard $\bar{\partial}$ -argument to the defining functions of the support functions. Then we get new support functions which are well-defined in a possibly narrow Stein neighborhood of $\bar{\Omega}$.
2. Asking for more might mean that we really want *globally* defined support surfaces, *i.e.*, support surfaces which are closed smooth complex hypersurfaces in \mathbb{C}^n , touching $b\Omega$ only from the outside at one distinguished boundary point. It is clear that this requires a much stronger hypothesis on the domain. Namely, we will assume that the given domain has a Runge neighborhood basis and is locally convexifiable of finite type near 0. It is one of the main results of this article (Theorem 1.3) that such closed global support surfaces then always do exist. Under suitable regularity assumptions on $b\Omega$ (namely $b\Omega$ has to be \mathcal{C}^∞ -smooth) smooth \mathcal{C}^∞ -families of such supporting hypersurfaces do indeed exist (Theorem 1.4).

In this part of the work we will prove the following statement:

Theorem 1.3 *Assume in addition to the hypotheses in Theorem 1.1 that $\bar{\Omega}$ has a Runge and Stein neighborhood. Then the map f can be chosen as a global automorphism of \mathbb{C}^n . A special case of this are convex domains of finite 1-type.*

Finally, in the case of bounded and smooth convex domains, we prove a version of Theorem 1.1 with parameters:

Theorem 1.4 *Let $\Omega \subset \mathbb{C}^n$ be a smooth and bounded convex domain of finite type $2k$. There exists a smooth family $\psi_\zeta \in \text{Aut}_{\text{hol}} \mathbb{C}^n$, $\zeta \in b\Omega$, such that $\psi_\zeta(\zeta)$ is a globally exposed $2k$ -convex boundary point for the domain $\psi_\zeta(\Omega)$.*

The structure of the article is as follows: In Section 2 we recall some local properties of convexifiable domains due to the two first authors. In Section 3 we prove Theorem 1.1. In Section 4 we prove Andersén-Lempert theorems with

parameters needed to prove Theorem 1.4, which we will do in Section 5. Finally, in Section 6, we give a brief sketch of how to prove Theorem 1.3 based on the arguments in Sections 3 and 5.

2 Local properties of convexifiable domains

Let Ω be a bounded \mathcal{C}^∞ -smooth domain in \mathbb{C}^n . In this section we recall the main facts about supporting hypersurfaces constructed in [4]. For this we suppose that there is an open set $V \subset \mathbb{C}^n$ such that $b\Omega \cap V$ is convex. Near any point $\zeta_0 \in b\Omega \cap V$ there is an open neighborhood V_{ζ_0} of ζ_0 , and a choice of a \mathcal{C}^∞ -family of coordinate changes $\{l_\zeta(z) : \zeta \in b\Omega \cap V_{\zeta_0}\}$ composed of a translation and a unitary transformation, such that, for each $\zeta \in b\Omega \cap V_{\zeta_0}$, $l_\zeta(\zeta) = 0$ and the unit outward normal vector n_ζ at ζ is turned by l_ζ into the unit vector $(1, 0, \dots, 0)$. In particular, $T_\zeta^\mathbb{C}b\Omega$ becomes in the new coordinates $\tilde{z} = l_\zeta(z)$ associated to ζ just $\{\tilde{z}_1 = 0\}$. The following is proved in [4]:

Theorem 2.1 *In the situation just described, assume that $b\Omega \cap V$ is of finite 1-type $2k$, and let $\tilde{V} \subset\subset V$. Then there exists a function $\hat{S}(\zeta, z) \in \mathcal{C}^\infty((b\Omega \cap \tilde{V}) \times \mathbb{C}^n)$, and constants $r, c > 0$, such that the following holds: for any choice of coordinate changes l_ζ as above, the function $S(\zeta, z) := \hat{S}(\zeta, l_\zeta^{-1}(z))$ is equal to*

$$S_\zeta(z) = 3z_1 + Kz_1^2 + g_\zeta(z'), \text{ where } (z_1, z') \text{ are coordinates on } \mathbb{C}^n, \quad (1)$$

and satisfies the estimate

$$\operatorname{Re} S_\zeta(z) \leq -c(|z_1|^2 + \|z'\|^{2k}), \quad (2)$$

for all $z \in B_r \cap l_\zeta(\Omega)$.

Note that if the domain Ω is convex, we get a \mathcal{C}^∞ -smooth function $\hat{S}(\zeta, z)$ on $b\Omega \times \mathbb{C}^n$.

3 The proof of Theorem 1.1

The proof of Theorem 1.1 is reduced to the two Lemmas in this section, Lemma 3.1 and Lemma 3.3.

We let e_1, \dots, e_n be the standard basis for the complex vector space \mathbb{C}^n and put $f_j := i \cdot e_j$ so that $e_1, f_1, \dots, e_n, f_n$ is a real basis. We denote the coordinates on \mathbb{C}^n by $z_j = x_{2j-1} + ix_{2j}$, we let \mathbb{C}_n denote the complex line $\mathbb{C}_n = \{z \in \mathbb{C}^n : z_1 = \dots = z_{n-1} = 0\}$ and we let π_n be the orthogonal projection to \mathbb{C}_n .

Our proof uses a technique from [8] invented for exposing points on a bordered Riemann surface in order to produce a proper holomorphic embedding (see also [6] Sections 8.8 and 8.9). We suppose that Ω is convexifiable near some point p on its boundary. Then we get the following situation:

Lemma 3.1 *For any $p \in b\Omega$ there exists $\Phi \in \text{Aut}_{\text{hol}}\mathbb{C}^n$ such that the following hold*

1. $\Phi(p) = 0$ and $T_0(b\Phi(\Omega)) = \{x_{2n} = 0\}$
2. The outward normal to $b\Phi(\Omega)$ at the origin is f_n ,
3. Near the origin we have that $b\Phi(\Omega)$ is k -convex at the origin in the following sense: The domain $\overline{\Phi(\Omega)} \subset \{z \in \mathbb{C}^n : x_{2n} - f(z', x_{2n-1}) \leq 0\}$ with $f(z', x_{2n-1}) \geq c(\|z'\|^{2k} + x_{2n-1}^2)$, $c > 0$ and
4. $\overline{\Phi(\Omega)} \cap \{z \in \mathbb{C}^n : z_1 = \dots = z_{n-1} = x_{2n-1} = 0, x_{2n} \geq 0\} = \{0\}$

Definition 3.2 *When condition (3) is satisfied near the origin we will refer to the origin as a strictly $2k$ -convex boundary point.*

Proof: It follows by Corollary 2.4 in [4] that there exists an open neighborhood U_p of p and an injective holomorphic map $\psi : U_p \rightarrow \mathbb{C}^n$ such that $\psi(\Omega \cap U_p)$ satisfies (1)-(3). Choosing an appropriate neighborhood $V_p \subset U_p$ of p , it follows that ψ is approximable by automorphisms ϕ of \mathbb{C}^n uniformly on V_p (see Section 4) and that $\psi(\Omega)$ is strictly k -convex near $\Phi(p)$ if ϕ is close enough to ψ . We proceed to achieve (4). Let

$$\Gamma := \{z \in \mathbb{C}^n : z_1 = \dots = z_{n-1} = x_{2n-1} = 0, x_{2n} \geq 0\}, \quad (3)$$

and let

$$\Gamma_0 := \{z \in \mathbb{C}^n : z_1 = \dots = z_{n-1} = x_{2n-1} = 0, 0 \leq x_{2n} \leq 1\}. \quad (4)$$

Choose an $R > 0$ such that $\overline{\Omega} \subset \mathbb{B}_R^n$. By [7] there exists $\psi_1 \in \text{Aut}_{\text{hol}}(\mathbb{C}^n)$ such that $\psi_1(z) = z + O(\|z\|^{2k+1})$ as $z \rightarrow 0$, such that $\psi_1(\Gamma_0) \cap \overline{\Omega} = \{0\}$, and such that $\psi_1(q) \in \mathbb{C}^n \setminus \overline{\mathbb{B}_R^n}$, where q denotes the endpoint of Γ_0 other than the origin. Consider the set $\psi_1^{-1}(\overline{\mathbb{B}_R^n}) \cap \Gamma \subset \Gamma \setminus \{q\}$. Since $\psi^{-1}(\overline{\mathbb{B}_R^n})$ is polynomially convex we have that $(\mathbb{C}^{n-1} \times \{i\}) \setminus \psi_1^{-1}(\overline{\mathbb{B}_R^n})$ is connected, and so using Weierstrass approximation theorem, we may construct a holomorphic shear map $\psi_2(z) = (z_1 + f_1(z), \dots, z_{n-1} + f_{n-1}(z), z_n)$ such that ψ_2 is close to the identity on Γ_0 , tangent to the identity to order $2k+1$ at the origin, and therefore not destroying strict k -convexity at 0, and such that $\psi_2(\overline{(\Gamma \setminus \Gamma_0)}) \cap \psi_1^{-1}(\overline{\mathbb{B}_R^n}) = \emptyset$. So $(\psi_1 \circ \psi_2)(\Gamma) \cap \overline{\Omega} = \emptyset$, and we set $\Phi = (\psi_1 \circ \psi_2)^{-1}$.

Lemma 3.3 *Let $W \subset \mathbb{C}^n$ be a bounded domain with $0 \in bW$ and assume that the following hold*

- (i) \overline{W} has a Stein neighborhood basis,
- (ii) W is strictly k -convex near the origin,
- (iii) $\overline{W} \cap \Gamma_0 = \{0\}$ with Γ_0 defined as in the proof of Lemma 3.1.

Then for any open set \tilde{V} containing Γ_0 and any small enough open set $V \subset \tilde{V}$ containing the origin, there exist a sequence of holomorphic embeddings $f_j : \overline{W} \rightarrow \mathbb{C}^n$ such that the following holds

- (1) $f_j \rightarrow id$ uniformly on $\overline{W} \setminus V$ as $j \rightarrow \infty$
- (2) $f_j(V) \subset \tilde{V}$ for all j ,
- (3) $f_j(0) = f_n$ for all j ,
- (4) $\text{Im}(\pi_n(f_j(z))) < 1$ for all $z \in (\overline{W} \cap V) \setminus \{0\}$ and
- (5) $f_j(W)$ is strictly k -convex at $f_j(0)$.

Proof: We let $\tilde{\Omega} \subset \mathbb{C}_n$ denote the domain $\tilde{\Omega} := \{z_n \in \Delta_\varepsilon : x_{2n} < f(0, \dots, 0, x_{2n-1})\}$. For some small $\delta > 0$ we define the following sets: $A := \{z \in \overline{W} \cap \overline{\mathbb{B}_\varepsilon^n} : x_{2n} \geq -\delta\}$ and $B := \{z \in \overline{W} \cap \overline{\mathbb{B}_\varepsilon^n} : x_{2n} \leq -\frac{\delta}{2}\} \cup \overline{W} \setminus \mathbb{B}_\varepsilon^n$. Then $A \setminus B \cap B \setminus A = \emptyset$ (if δ is small) and by Theorem 4.1 in [5], for any open set \tilde{C} containing the set $C := A \cap B$ there exist open sets A', B', C' with $A \subset A'$, $B \subset B'$ and $C \subset C' \subset A' \cap B' \subset \tilde{C}$, such that if $\gamma : \tilde{C} \rightarrow \mathbb{C}^n$ is injective holomorphic, and sufficiently close to the identity, then there exist holomorphic injections $\alpha : A' \rightarrow \mathbb{C}^n$, $\beta : B' \rightarrow \mathbb{C}^n$, uniformly close to the identity on their respective domains (depending on γ), and such that

$$\gamma = \beta \circ \alpha^{-1} \text{ on } C'. \quad (5)$$

(This can also be found in Theorem 8.7.2, page 359 in [6].) Choose a simply connected smooth domain $U \subset \mathbb{C}_n$ with $\pi_n(A) \subset U$ and such that near the origin $U = \{z \in \mathbb{C}_n : x_{2n} < 0\}$. For $j \in \mathbb{N}$ let l_j denote the line segment $l_j = \{z_n \in \mathbb{C}_n : x_{2n-1} = 0, 0 \leq x_{2n} \leq 1/j\}$. For each j it follows from Mergelyan's Theorem that we may choose injective holomorphic maps $\sigma_j : \overline{U} \cup l_j \rightarrow \mathbb{C}_n$ such that σ_j approximately stretches l_j to cover Γ_0 such that $\sigma_j(z) = (1 - 1/j)i + z + O(|z - i/j|)^{2k+1}$ and such that $\sigma_j \rightarrow id$ on \overline{U} as $j \rightarrow \infty$. For each j let U_j be a domain obtained from U by adding a strip around l_j of width less than $1/j$ which is then smoothened and made strictly convex at the end point l_j . U_j should lie inside where σ_j is injective holomorphic, and be chosen such that $\sigma_j(U_j)$ is strictly convex near the end point of $\sigma_j(l_j) = f_n$ and such that $\text{Im}(\sigma_j(z_n)) < 1$ for all $z \in \overline{U_j} \setminus \frac{1}{j}f_n$. Let ψ_j be a holomorphic diffeomorphism from \overline{U} to $\overline{U_j}$ such that $\psi_j(0) = \frac{i}{j}$, $\psi_j \rightarrow id$ uniformly on \overline{U} . (See Goluzin, [10], Theorem 2, p. 59.) Let $\phi_j = \sigma_j \circ \psi_j$ and let γ_j be an extension of ϕ_j to A . Then $\text{Im}(\Pi_n(\gamma_j(z))) < 1$ for all $z \in A \setminus \{0\}$. It is not hard to see that $\gamma_j(A)$ is strictly k -convex near f_n and $\gamma_j \rightarrow id$ on a neighborhood of C . We get splittings

$$\gamma_j \circ \alpha_j = \beta_j \quad (6)$$

as explained above. If j is large enough, we get that (7) defines an injective holomorphic map f_j on $\tilde{\Omega}$, and if α_j is close enough to the identity, since α_j can be assumed to vanish to order $2k+1$ at the origin, we get that $\text{Im}(f_j(z)) < 1$ for all $z \in A \setminus \{0\}$ and such that $f_j(A)$ is strictly k -convex at $f_j(0)$.

4 Andersén-Lempert with parameters in a smooth manifold, and approximation with jet interpolation.

A parameter version of the Andersen-Lempert theorem [1] for holomorphic parameters was proved by Kutzschebauch [11]. Jet interpolation results without parameters have been proved by Forstnerič [9] and Weickert [13] (see also sections 4.9 and 4.15 in [6]). For a smooth manifold M we let (ζ, z) denote the coordinates on $M \times \mathbb{C}^n$. For any $\zeta \in M$ we denote by \mathbb{C}_ζ^n the slice $\{\zeta\} \times \mathbb{C}^n$, and for any subset $\Sigma \subset M \times \mathbb{C}^n$ we let Σ_ζ denote the slice $\Sigma_\zeta := \mathbb{C}_\zeta^n \cap \Sigma$.

Theorem 4.1 *Let M be a compact smooth manifold and let $\Omega \subset M \times \mathbb{C}^n$ be a domain, $n \geq 2$. Let $K \subset \Omega$ be a compact set, and let $\phi : [0, 1] \times \Omega \rightarrow M \times \mathbb{C}^n$ be a C^2 -smooth map such that, writing $\phi(t, \zeta, z) = \phi_t(\zeta, z)$, the following hold*

- (1) $\phi_t(\zeta, z) = (\zeta, \varphi_t(\zeta, z)) = (\zeta, \phi_{t,\zeta}(z))$,
- (2) $\phi_{t,\zeta} : \Omega_\zeta \rightarrow \mathbb{C}_\zeta^n$ is injective holomorphic, and
- (3) $K_{t,\zeta} := \phi_{t,\zeta}(K_\zeta)$ varies continuously with (t, ζ) and is polynomially convex

for all $t \in [0, 1], \zeta \in M$.

Then ϕ_1 is uniformly approximable on K by a smooth family $\psi(\zeta, z)$ with $\psi_\zeta \in \text{Aut}_{\text{hol}} \mathbb{C}_\zeta^n$ if (and only if) ϕ_0 is approximable by such a family. Moreover, if (1)–(3) hold and if $a(\zeta) \in K_\zeta^\circ$ is a smoothly parametrized family of points, and if $d \in \mathbb{N}$, we may additionally achieve that

- (4) $\phi_{1,\zeta}(z) - \psi_\zeta(z) = O(\|z - a(\zeta)\|^d)$, as $z \rightarrow a(\zeta)$.

Proof: We give a sketch of the proof of the first claim; the point is just to verify that the non-parametric proof goes through without change with parameters. The assumption that ϕ_0 is approximable allows us to assume $\phi_0 = \text{id}$. Define first a parametrized vector field

$$X_{t,\zeta}(\phi_{t,\zeta}(z)) := \frac{d}{dt} \phi_{t,\zeta}(z). \quad (7)$$

Then $X_{t,\zeta}$ is an inhomogeneous vector field, holomorphic in z , whose flow is $\phi_{t,\zeta}(z)$. For each t let $\varphi_{t,\zeta}^s$ denote the time- s flow of the homogenous vector field $X_{t,\zeta}$ where t is fixed. It is well known that there is a partitioning $[j/n, (j+1)/n], j = 0, \dots, n-1$ of $[0, 1]$, such that the composition

$$\varphi_{(n-1)/n,\zeta}^{1/n} \circ \dots \circ \varphi_{0,\zeta}^{1/n} \quad (8)$$

approximates $\phi_{\zeta,1}$ on K . So the problem is reduced to approximating the flow φ_ζ^1 of a homogenous vector field X_ζ on a family K_ζ .

Next, by assumption (3) and approximation, we may assume that X_ζ is a polynomial vector field

$$X_\zeta(z) = \sum_{j=1}^N g_j(\zeta) X_j(z), \quad (9)$$

with coefficients g_j in $\mathcal{E}(M)$; this can be obtained by gluing a fiberwise Runge-approximation using a partition of unity on M . Now the main point of Andersén-Lempert Theory in \mathbb{C}^n is that *any m -homogenous polynomial vector field V_m is a sum of completely integrable vector fields* (see e.g. [6], Lemma 4.9.5):

$$V_m(z) = \sum_{i=1}^r c_i \lambda_i(z)^m \cdot v_i + d_i \lambda_i(z)^{m-1} \langle z, v_i \rangle \cdot v_i, \quad (10)$$

with $c_i, d_i \in \mathbb{C}$, $v_i \in \mathbb{C}^n$ and $\lambda_i \in (\mathbb{C}^n)^*$ with $\lambda_i(v_i) = 0$. The flows of these two types of vector fields are

$$z \xrightarrow{f_{t,j}} z + t \cdot c_i \lambda_i(z)^m \cdot v_i \text{ and } z \xrightarrow{g_{t,j}} z + (e^{td_i \lambda_i(z)^m} - 1) \langle z, v_i \rangle \cdot v_i. \quad (11)$$

Applying this to each of the vector fields $X_j(z)$ in (9) we get that

$$X_\zeta(z) = \sum_{j=1}^{\tilde{N}} \tilde{g}_j(\zeta) \cdot \tilde{X}_j(z), \quad (12)$$

where each \tilde{X}_j is completely integrable with flow ψ_j^s , and so X_ζ is a sum of completely integrable fields with flows $\psi_{\zeta,j}^s = \psi_j^{g(\zeta) \cdot s}$. Finally the sequence

$$(\psi_{\zeta,N}^{1/n} \circ \dots \circ \psi_{\zeta,1}^{1/n})^n \quad (13)$$

converges uniformly to φ_ζ^1 as $n \rightarrow \infty$.

Finally we consider (4). We will correct the initial approximation at $a(\zeta)$ and by translation we may assume that $a(\zeta) = 0$ for all ζ , and that both ϕ and ψ fix the origin. Define $J_{d-1}(\zeta)$ to be the $(d-1)$ -jet of $\psi_\zeta^{-1} \circ \phi_{1,\zeta}$. It is easy to see that we may assume that $J_{d-1}(z) = id + h.o.t$, and by the Cauchy estimates we may assume that $J_{d-1}(\zeta)$ is arbitrarily close to the identity map. We will correct ψ_ζ inductively, and our induction assumption is that $J_{d-1}(\zeta) = O(\|z\|^m)$, $2 \leq m \leq d-1$.

Using (10) we fix an expansion

$$z^\alpha \cdot e_j = s_{\alpha,j}(z) := \sum_{i=1}^r c_i^{\alpha,j} \lambda_i^{\alpha,j}(z)^m \cdot v_i^{\alpha,j} + d_i^{\alpha,j} \lambda_i^{\alpha,j}(z)^{m-1} \langle z, v_i^{\alpha,j} \rangle \cdot v_i \quad (14)$$

for each multi-index $|\alpha| = m$ and $j = 1, \dots, n$. Now expand the m -homogenous part $J_{d-1,m}$ of J_{d-1} using (14)

$$J_{d-1,m}(\zeta) = \sum_{|\alpha|=m, 1 \leq j \leq n} h_{\alpha,j}(\zeta) \cdot s_{\alpha,j}(z). \quad (15)$$

It is easy to see that the composition Φ_m of all automorphisms

$$z \mapsto z + h_{\alpha,j}(\zeta) \cdot c_i^{\alpha,j} \lambda_i^{\alpha,j}(z)^m \cdot v_i^{\alpha,j} \quad (16)$$

and

$$z \mapsto z + (e^{h_{\alpha,j}(\zeta)d_i^{\alpha,j}\lambda_i^{\alpha,j}(z)^m} - 1)\langle z, v_i^{\alpha,j} \rangle \cdot v_i^{\alpha,j}. \quad (17)$$

matches $J_{d-1,m}$ to order m , and we may assume that Φ_m is as close to the identity as we like on a compact set since all the $h_{\alpha,j}$'s can be assumed to be as small as we like. It follows that the map $\psi_\zeta \circ \Phi_m$ is a small perturbation of ψ_ζ which matches $\phi_{1,\zeta}$ to order m . The induction step is complete.

Remark 4.2 *For a more detailed explanation of jet-completion (without parameters) the reader can consult [6] page 154–158.*

5 The construction with parameters: Proof of Theorem 1.4

Theorem 5.1 *Let $\Omega \subset \mathbb{C}^n$ be a smooth and bounded convex domain of finite type $2k$. There exists a smooth parameter family $\psi_\zeta \in \text{Aut}_{\text{hol}}\mathbb{C}^n$, $\zeta \in b\Omega$, such that $\psi_\zeta(\zeta)$ is a globally exposed $2k$ -convex boundary point for the domain $\psi_\zeta(\Omega)$.*

Proof: By [4] there exist $r, c > 0$ and a smooth parameter family

$$\psi_\zeta(z) \text{ defined on } \{(\zeta, z) : \|z - \zeta\| < r\} \quad (18)$$

such that $\psi(\zeta, \cdot)$ is injective holomorphic for all ζ and the following holds for all ζ (see Section 2): let n_ζ denote the outward pointing unit normal vector to $b\Omega$ at ζ , let l_ζ be a composition of a translation and a unitary transformation such that $l_\zeta(\zeta) = 0$ and such that n_ζ is sent to the vector $(1, 0, \dots, 0)$. Then $\tilde{\psi}_\zeta(z) := \psi_\zeta \circ l_\zeta^{-1}$ is of the form $(S_\zeta(z), z_2, \dots, z_n)$, and S satisfies

$$S_\zeta(z) = 3z_1 + Kz_1^2 + g_\zeta(z'), \quad z = (z_1, z'), \quad (19)$$

(See Section 2.) Moreover, we have that

$$\text{Re}(S_\zeta(z)) \leq -c \cdot (|z_1|^2 + \|z'\|^{2k}), \quad z \in B_r(\zeta) \cap \overline{\Omega}, \quad (20)$$

where the constant $c > 0$ does not depend on ζ .

Our first step is to change the maps ψ_ζ conveniently on the normals n_ζ , and then approximate the changed maps by a family of holomorphic automorphisms. Set $\Gamma_0 := \{z \in \mathbb{C}^n : 0 \leq x_1 \leq 1, x_2 = z_2 = \dots = z_n = 0\}$, and let h denote the map $h(z) = 3z_1 + Kz_1^2$ near Γ_0 . By changing h smoothly, then finding a smooth homotopy of maps, and finally applying Mergelyan's Theorem with parameters, we find $\delta > 0$ and a smooth map

$$\tilde{h} : [0, 1] \times \Gamma_0(\delta) \rightarrow \mathbb{C},$$

such that the following hold

- (1) $\tilde{h}_0(z_1) = 3z_1$,
- (2) $\tilde{h}_t(\cdot)$ is injective holomorphic for each $t \in [0, 1]$,

- (3) $\tilde{h}_1(z_1) \approx h(z_1)$ on $B_\delta(0)$ and $(\tilde{h}_1 - h)(z) = O(|z|^{2k+1})$ as $z \rightarrow 0$,
- (4) $\tilde{h}_1(z) \approx 3z_1$ on $B_\delta(1)$ and $(\tilde{h}_1 - 3)(z) = O(|z - 1|^{2k+1})$ as $z \rightarrow 1$,
- (5) $\tilde{h}_1^{-1}(3\Gamma_0) \approx \Gamma_0$.

We define a homotopy modification $\hat{\psi}_{\zeta,t}(z)$ of ψ_ζ by setting

$$\hat{S}_{\zeta,t}(z) := \tilde{h}_t(z_1) + t \cdot g_\zeta(z') \text{ on } \Gamma_0(\delta). \quad (21)$$

in local coordinates.

Let $b(\zeta)$ denote the end point of n_ζ other than ζ , and note that by Stolzenberg [12] we may assume, by possibly having to decrease δ , that

$$\hat{K}_{\zeta,t} := \hat{\psi}_{\zeta,t}(\overline{B_\delta(\zeta) \cup n_\zeta \cup B_\delta(b(\zeta))})$$

is polynomially convex for all ζ . By Theorem 4.1 and its proof there exist families $G_\zeta, H_\zeta \in \text{Aut}_{\text{hol}} \mathbb{C}^n$ such that the following holds

- (6) $G_\zeta \approx \hat{\psi}_{\zeta,1}$ on $B_\delta(\zeta)$, and $(G_\zeta - \hat{\psi}_\zeta)(z) = O(\|z - \zeta\|^{2k+1})$ as $z \rightarrow \zeta$,
- (7) $H_\zeta \approx \hat{\psi}_{\zeta,1}^{-1}$ on $B_\delta(0) \cup 3n_\zeta \cup B_\delta(3b(\zeta))$,
- (8) $(H_\zeta - \hat{\psi}_{\zeta,1}^{-1})(z) = O(\|z - 3b(\zeta)\|^{2k+1})$ as $z \rightarrow 3b(\zeta)$, and
- (9) $H_\zeta \circ G_\zeta \approx \text{id}$ on $\overline{\Omega}$.

Next we construct a *continuous* parameter family of exposing maps f_ζ as in Lemma 3.3, where each f_ζ wraps the boundary at $G_\zeta(\zeta)$ around the normal $3n_\zeta$. The composition $H_\zeta \circ f_\zeta \circ G_\zeta$ will globally expose the point ζ $2k$ -convexly. We will then change f_ζ to depend *smoothly* on ζ , and in a final step we will approximate the family f_ζ by a smooth family of automorphisms.

Choose a strictly pseudoconvex neighborhood Ω' of $\overline{\Omega}$ close to Ω and let ρ be a smooth strictly plurisubharmonic defining function for Ω' near $b\Omega'$. For $0 < r \ll 1$ we let $\Omega'(r) := \{z : \rho(z) < r\}$. For $0 < \sigma \ll 1$ we define Cartan pairs $\tilde{A}_\zeta(r) := \Omega'(r) \cap \overline{B_\sigma(\zeta)}$, $\tilde{B}_\zeta(r) := \Omega'(r) \setminus B_{\sigma/2}(\zeta)$. Set $A_\zeta(r) := G_\zeta(\tilde{A}_\zeta(r))$, $B_\zeta(r) := G_\zeta(\tilde{B}_\zeta(r))$.

Let γ_j be the sequence of locally exposing maps from the proof of Lemma 3.3. Since the maps only depend on the normal coordinate, the map $\tilde{\gamma}_{j,\zeta} := l_\zeta^{-1} \circ \gamma_j \circ l_\zeta$ is a well defined family of locally exposing maps for Ω , and $\tilde{\gamma}_{j,\zeta} \rightarrow \text{id}$ uniformly on $C_\zeta(r) := A_\zeta(r) \cap B_\zeta(r)$ for small enough r independently of ζ . To globalize these locally defined maps we use the following parametric version of Theorem 8.7.2 in [6].

Lemma 5.2 *If r_0 is small enough and $\mu > 0$ there exist $r_1 < r_0$ and $\epsilon > 0$ such that the following holds: for any family $\gamma_\zeta : \overline{C_\zeta(r_0)} \rightarrow \mathbb{C}^n$ of holomorphic maps with $\|\gamma_\zeta - \text{id}\|_{\overline{C_\zeta(r_0)}} < \epsilon$, continuous in ζ , there exist injective holomorphic maps $\alpha_\zeta : A_\zeta(r_1) \rightarrow \mathbb{C}^n$, $\beta_\zeta : B_\zeta(r_1) \rightarrow \mathbb{C}^n$, continuous in ζ , such that*

$$\gamma_\zeta = \beta_\zeta \circ \alpha_\zeta^{-1}, \|\alpha_\zeta - \text{id}\|_{A_\zeta(r_1)} < \mu, \|\beta_\zeta - \text{id}\|_{B_\zeta(r_1)} < \mu. \quad (22)$$

Moreover, we may achieve that $(\alpha_\zeta - \text{id})(z) = O(\|z - \zeta\|^{2k+1})$ as $z \rightarrow \zeta$.

The proof of this is almost identical to that in [6], noting that there exist a solution operator to the $\bar{\partial}$ -equation which is continuous with parameters, and one can multiply by powers of S_ζ to get exact jet interpolation.

So if j is chosen large enough we get that the family f_ζ defined as $f_\zeta := \gamma_{\zeta,j} \circ \alpha_{j,\zeta}$ on $A_\zeta(r_1)$ and $f_\zeta := \gamma_{\zeta,j} \circ \beta_{\zeta,j}$ on $B_\zeta(r_1)$, is a family of injective holomorphic maps $\tilde{\gamma}_\zeta : G_\zeta(\Omega'(r_1)) \rightarrow \mathbb{C}^n$ exposing the point ζ $2k$ -convexly. By (5) and (8) the family $H_\zeta \circ f_\zeta \circ G_\zeta$ is a continuous family of holomorphic injections on $\Omega'(r_1)$, globally exposing the point ζ for the domain Ω .

Next we approximate f_ζ by a *smooth* family of exposing maps. This is done using a partition of unity on $b\Omega$. Note first that although f_ζ is only continuous in ζ , the $2k$ -jet at ζ , $J(\zeta)$, is smooth in ζ ; this is because $\alpha_{j,\zeta}$ vanishes to order $2k$ at ζ . Let $(U_j, \alpha_j), j = 1, \dots, m$, be a partition of unity on $b\Omega$ with a point $a_j \in U_j$ for all j . For each j write $f_{a_j}(z) = z + g_j(z)$. We set $\tilde{f}_\zeta(z) := z + \sum_{j=1}^m \alpha_j(\zeta) g_j(z)$. By choosing the covering fine enough we may achieve that \tilde{f}_ζ is as close to f_ζ as we like on $\bar{\Omega}$, and also that the $2k$ -jet of \tilde{f}_ζ at ζ is as close to that of f_ζ as we like. So using the argument in the proof of Theorem 4.1 we can correct \tilde{f}_ζ so that its $2k$ -jet at ζ matches that of f_ζ exactly.

Finally we need to approximate the family \tilde{f}_ζ by a family of automorphisms. We may assume that $0 \in \Omega, G_\zeta(0) = 0$, and that $f_\zeta(0) = 0$ for all ζ . Set

$$\varphi_t(\zeta, z) := G_\zeta\left(\frac{1}{t}\tilde{f}_\zeta(t \cdot G_\zeta^{-1}(z))\right). \quad (23)$$

We may assume that $\tilde{f}_\zeta(G_\zeta(\bar{\Omega}))$ is polynomially convex for all $\zeta \in b\Omega$. In that case it follows that there exists some $s > 1$ such that $\varphi_{t,\zeta}(G_\zeta(s\bar{\Omega}))$ is polynomially convex for all t, ζ , and so approximation follows by Theorem 4.1. It is enough to show that $f_\zeta(G_\zeta(\bar{\Omega}))$ is polynomially convex.

Fix $\zeta \in b\Omega$. By Stolzenberg [12] we have that $G_\zeta(\bar{\Omega}) \cup 3n_\zeta$ is polynomially convex. Let W_ζ be a Runge neighborhood of $K_\zeta := G_\zeta(\bar{\Omega}) \cup 3n_\zeta$, very close to K_ζ . Consider a point $b \in b\Omega \cap \bar{B}_\zeta(0)$. If W_ζ is close enough to K_ζ , and if β_ζ is close enough to the identity, then the locally defined function $e^{C \cdot S_b(\beta_\zeta^{-1}(z))}$ for $C \gg 0$ may be approximately globalized to W_ζ , separating points on $\beta_\zeta(n_b)$ close to $\beta_\zeta(b)$ from $f_\zeta(G_\zeta(\bar{\Omega}))$ as long as f_ζ is chosen such that $f_\zeta(\bar{\Omega}) \subset W_\zeta$. It follows that

$$cl[\widehat{f_\zeta(G_\zeta(\bar{\Omega}))} \setminus f_\zeta(G_\zeta(\bar{\Omega}))] \cap f_\zeta(G_\zeta(\bar{\Omega})) \subset f_\zeta(A_\zeta(0)). \quad (24)$$

Hence by Rossi's local maximum principle

$$\widehat{f_\zeta(G_\zeta(\bar{\Omega}))} = f_\zeta(G_\zeta(\bar{\Omega})) \cup [\widehat{f_\zeta(A_\zeta(0))} \cap G_\zeta(\bar{\Omega})]. \quad (25)$$

But f_ζ^{-1} is approximable by entire maps on $f_\zeta(A_\zeta(0))$, and so $f_\zeta(G_\zeta(\bar{\Omega}))$ is polynomially convex.

6 Remark on the proof of Theorem 1.3

The proof of Theorem 1.3 is almost the same as that of Theorem 1.1, except that we need to make sure that the exposing maps f_j are approximable by

holomorphic automorphisms. To see why this is so, note first that each γ_j may be connected to the identity map by an isotopy which is uniformly close to the identity on C . The Cartan type splitting with parameters then allows us to construct each f_j as the time-1 map of an isotopy $f_{j,t}$ with $f_{j,0} = \text{id}$ (this argument allows us to avoid the usual assumption in Andersén-Lempert theory that Ω is star shaped). This isotopy is only \mathcal{C}^0 but we can obtain a smooth isotopy by gluing as before. The same argument as in the previous section tells us that we may assume that $f_{t,j}(\overline{\Omega})$ is polynomially convex for all t if j is sufficiently large, and so we may approximate by automorphisms.

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